MEM6810 Engineering Systems Modeling and Simulation
工程系统建模与仿真

Theory Analysis

Lecture 3: Queueing Models

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上海交通大学 SHANGHAI JIAO TONG UNIVERSITY | 董浩云智能制造与服务管理研究院 CY TUNG Institute of Intelligent Manufacturing and Service Management (中美物流研究院) (Sino-US Global Logistics Institute)

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   - Jackson Networks

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- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.

Figure: Queues in Hospital

Figure: Queues in Store (*from* The Sun)

Figure: Queues in Campus (for COVID-19 Nucleic Acid Test)

Figure: Queues in Bank

Figure: Queues in Bank (No requirement to *stand physically* in queues)

Figure: Queue in Online Service

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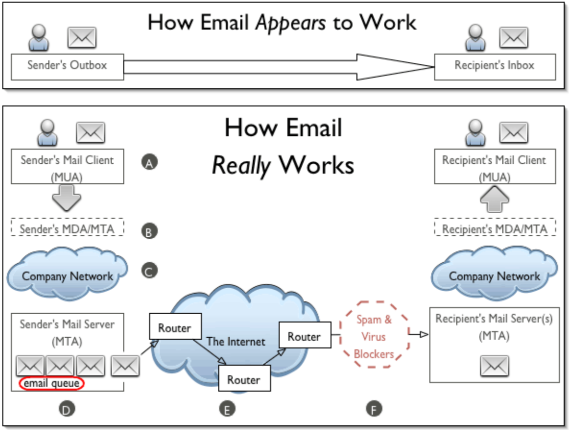

Figure: Queue in Mail Server (*from* OASIS)

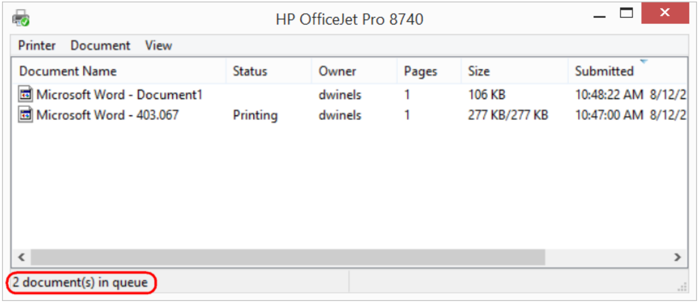

Figure: Queue in Printer

Figure: Queues (Inventories) in Manufacturing Line (*from* Estes)

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  - Manufacturing systems maintain queues (called inventories) of raw materials, partly finished goods, and finished goods via the manufacturing process.

- Typically, a queueing system consists of a stream of "**customers**" (humans, goods, messages) that
  - arrive at a service facility;
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- Queueing models are mathematical representation of queueing systems.

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- They help us do this by answering the following questions (and many others):
  1. How many customers are there in the queue (or station) on average?
  2. How long does a typical customer spend in the queue (or station) on average?
  3. How busy are the servers on average?

- *Simple queueing models solved analytically*:
  - Get rough estimates of system performance with negligible time and expense.
  - *More importantly, understand the dynamic behavior of the queueing systems and the relationships between various performance measures.*
  - Provide a way to verify that the simulation model has been programmed correctly.

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- *Complex queueing models analyzed through simulation*:
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- This lecture focuses on the classical analytically solvable queueing models.

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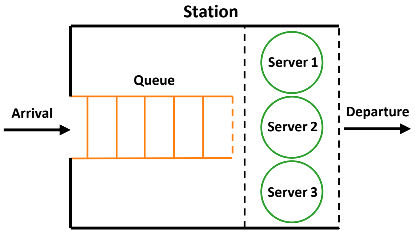
- Suppose that there is only **one queue** in one station.

- **Capacity** is the maximal number of customers allowed in the station.
  - Number waiting in queue + number having service.
  - Finite or infinite.

- Single-station queueing system.
  - Customers simply leave after service.
  - E.g., customers arrive to buy coffee and then leave.



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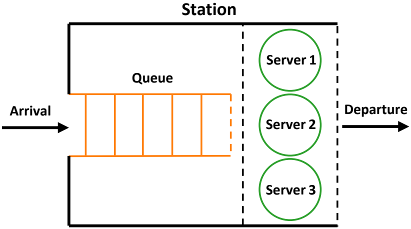
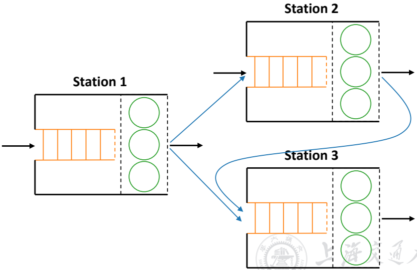
- Multiple-station queueing system (queueing network).
  - Customers can move from one station to another (for different service), before leaving the system.
  - E.g., patients wait and get service at several different units inside a hospital.





- The **arrival process** describes how the customers come.
  - Arrivals may occur at *scheduled* times or *random* times.
  - When at random times, the **interarrival times** are usually characterized by a probability distribution.
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- An customer arriving at a station:
  - if the station capacity is full:
    - the external arrival will leave immediately (called **lost**);
    - the internal arrival may wait in the previous station (may **block** the previous server).
  - if the station capacity is not full, enter the station:
    - if there is idle server in the station, get service immediately;
    - if all servers are busy, wait in the **queue**.

- Queue discipline: Which customer to serve first.
  - First-in-first-out (FIFO), or first-come-first-served (FCFS).
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- Queue behavior: Actions of customers while waiting.
  - Balk: leave when they see that the line is too long.
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- **Service time** is the duration of service in a server.
  - *Constant* or *random* duration.
  - May depend on the customer type.
  - May depend on the time of day or the queue length.

- When without specification, the queueing models considered in this lecture shall satisfy the following:
  1. One customer type.
  2. Random arrivals (i.e., random interarrival times, iid.).
  3. No batch (or say, batch size is 1).†
  4. One queue in one station.
  5. First-come-first-served (FCFS).
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- Even so, it is not that easy to analyze the queueing models!

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- Examples: $M/M/1$, $M/G/1$, $M/M/s/K$.

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4 Queueing Networks
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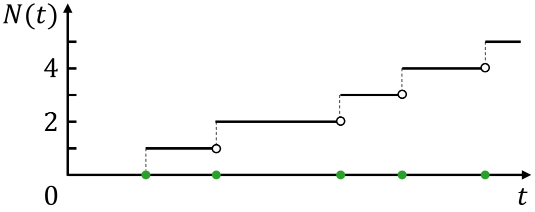

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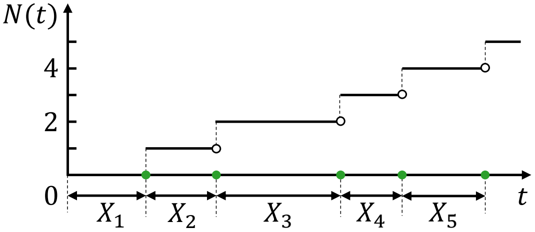

- Let $\{X_n,\ n \geq 1\}$ denote the *interarrival times*:
  - X_1 denotes the time of the first arrival;
  - For $n \geq 2$, X_n denotes the time between the $(n-1)$st and the nth arrivals.

- **Definition 1.** The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has **independent** and **stationary** increments;
  - For $t > 0$, $N(t) \sim \mathrm{Pois}(\lambda t)$, i.e.,

$$\mathbb{P}(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ \ n = 0, 1, 2, \ldots.$$

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- Stationary Increments: The distribution of number of arrivals in any time interval depends only on the length of time interval, i.e., for $s < t$, the distribution of $N(t) - N(s)$ depends only on $t - s$.

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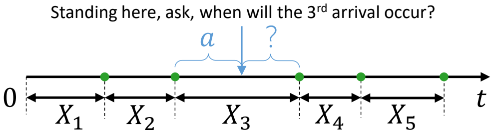
- **Definition 1**, **Definition 2** and **Definition 3** are equivalent.

- **Question 1**: When will the next appear?



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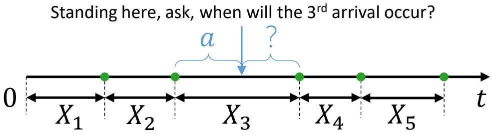

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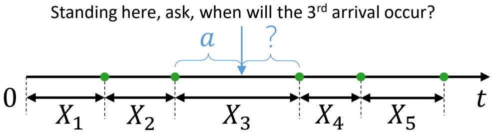

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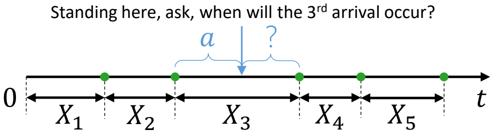

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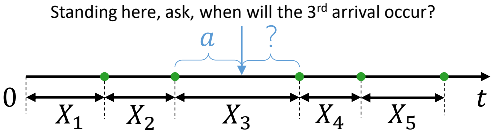

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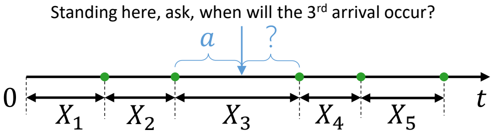

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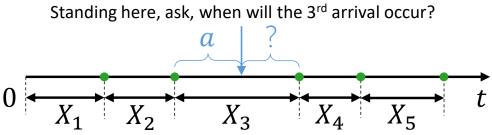

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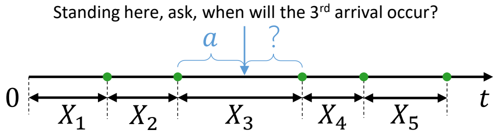
$$

- **Question 1**: When will the next appear?



$$\begin{aligned}
\mathbb{P}(X_3 - a > x | X_3 > a) &= \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x)}{\mathbb{P}(X_3 > a)} \\
&= \frac{e^{-\lambda(a+x)}}{e^{-\lambda a}} = e^{-\lambda x}. \quad \text{(Not related to } a\text{!)}
\end{aligned}$$

- The Poisson process has no memory! (equivalent to the independent and stationary increments assumption)

- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the arrival time of the nth arrival.
- **Question 2**: If I only know there are n arrivals up to time t, what can I say about the n arrival times $S_1, \ldots, S_n$?

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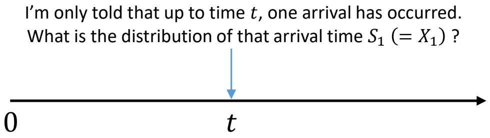

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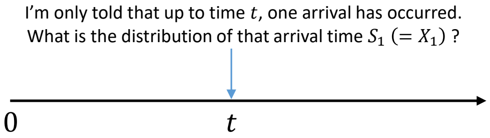

- Intuition:
  - Since Poisson process possesses independent and stationary increments, each interval of equal length in $[0, t]$ should have the same probability of containing the arrival.
  - Hence, the arrival time should be uniformly distributed on $[0, t]$.

Proof.

$\mathbb{P}\{X_1 < s | N(t) = 1\}$

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- Remark: This result can be generalized to n arrivals.

Property (Conditional Distribution of Arrival Times)

Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent RVs uniformly distributed on the interval $(0, t)$.

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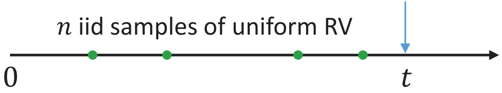

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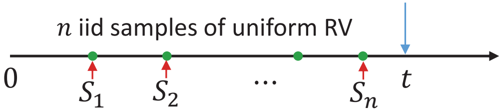

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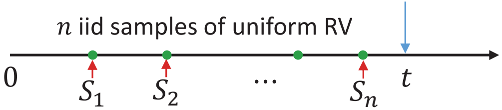

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- This is very nice for simulation!

1 Queueing Systems and Models
- Introduction
- Characteristics & Terminology
- Kendall Notation

2 Poisson Process
- Definition
- Properties

3 Single-Station Queues
- Notations
- General Results
- Little's Law
- $M/M/1$ Queue
- $M/M/s$ Queue
- $M/M/\infty$ Queue
- $M/M/1/K$ Queue
- $M/M/s/K$ Queue
- $M/G/1$ Queue

4 Queueing Networks
- Jackson Networks

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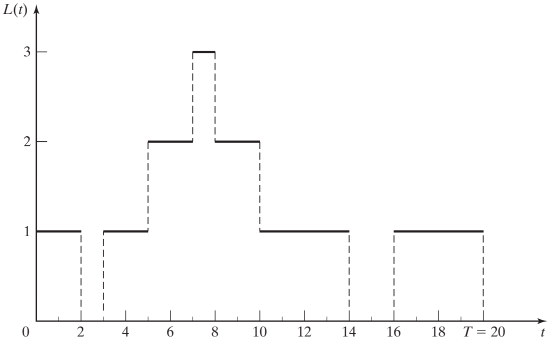

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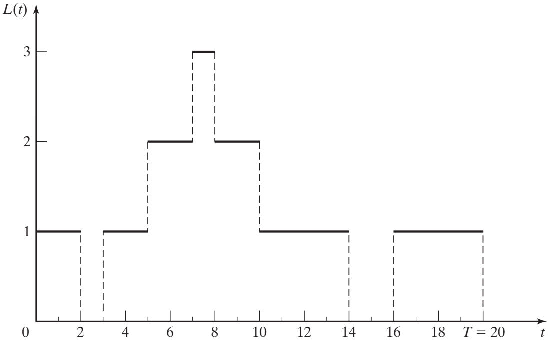
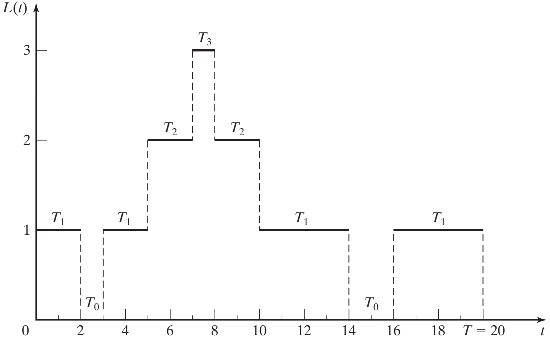


Figure: Illustration of $L(t)$ (*from Banks et al. (2010)*)

- Let $\widehat{L}(T)$ denote the (time-weighted) average number of customers in the station up to time T:

$$\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t.$$

- Another expression of $\widehat{L}(T)$: Let T_n denote the total time during $[0, T]$ in which the station contains exactly n customers.



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- $\widehat{L}(T) \coloneqq \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right).$

- Suppose during time $[0, T]$, totally $N(T)$ customers have entered the station, and let $W_1, W_2, \ldots, W_{N(T)}$ denote the time each customer spends in the station up to time T.[†]

[†] The time includes both the waiting time in queue and the time in server. The part after T is not counted.

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- Let $\widehat{W}(T)$ denote the average sojourn time (逗留时间) in the station up to time T:

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- In a similar way, we can also define
  - $\widehat{L}_Q(T)$ – The average number of customers in the *queue* up to time T.
  - $\widehat{W}_Q(T)$ – The average *waiting* time in the *queue* up to time T.

[†] The time includes both the waiting time in queue and the time in server. The part after T is not counted.

- Now we consider the long-run measures.
  - L – The long-run average number of customers in the station:
    $$L := \lim_{T\to\infty} \widehat{L}(T).$$
  - W – The long-run average sojourn time in the station:
    $$W := \lim_{T\to\infty} \widehat{W}(T).$$
  - L_Q – The long-run average number of customers in the queue:
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  - W_Q – The long-run average waiting time in the queue:
    $$W_Q := \lim_{T\to\infty} \widehat{W}_Q(T).$$
- Question: When will L, W, L_Q and W_Q exist (and $< \infty$)?

- We also define the *limiting probability* that there will be exactly n customers in the station as time goes to infinity:

$$P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \quad n = 0, 1, 2, \ldots.$$

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- Moreover, for an arbitrary $X/Y/s/K$ queue
  - Let λ denote the arrival rate, i.e.,

$$\mathbb{E}[\text{interarrival time}] = \frac{1}{\lambda}.$$

  - Let μ denote the service rate in one server, i.e.,

$$\mathbb{E}[\text{service time}] = \frac{1}{\mu}.$$

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- Question: When will L, W, L_Q, W_Q and P_n exist?

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Theorem 1 (Condition of Stability)

For an $X/Y/s/\infty$ queue (i.e., infinite capacity) with arrival rate λ and service rate μ, it is stable if

$$\lambda < s\mu.$$

And, an $X/Y/s/K$ queue (i.e., finite capacity) will always be stable.

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- P_n is also called the probability that there are exactly n customers in the station when it is in the *steady state*.
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  - Since the system is stable and run for infinitely long time, it should enters some steady state (i.e., has nothing to do with the initial state).
- L can also be written as $L := \sum_{n=0}^{\infty} nP_n$ (see next slide).
  - L is also called the expected number of customers in the station in steady state;
  - W is also called the expected sojourn time in the station in steady state;
  - L_Q is also called the expected number of customers in the queue in steady state;
  - W_Q is also called the expected waiting time in the queue in steady state.

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- It turns out that, when the queue is *stable*, P_n also equals the *long-run proportion of time that the station contains exactly* n *customers*,[†] i.e., with probability 1, for all n,

$$P_n = \lim_{T\to\infty} \frac{\text{amount of time during } [0,T] \text{ that station contains } n \text{ customers}}{T}.$$

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  - It is named after John D.C. Little, who was the first to prove a version of it, in 1961.
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Theorem 2 (Little's Law – Empirical Version)

Define the observed entering rate $\widehat{\lambda} := N(T)/T$, then

$$\widehat{L}(T) = \widehat{\lambda}\widehat{W}(T), \quad \widehat{L}_Q(T) = \widehat{\lambda}\widehat{W}_Q(T).$$

- Verify Little's Law.

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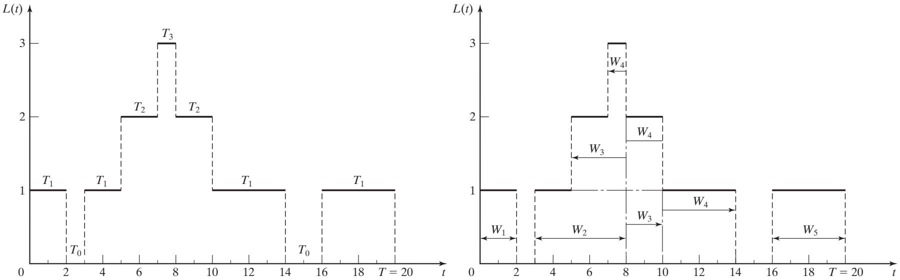

Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

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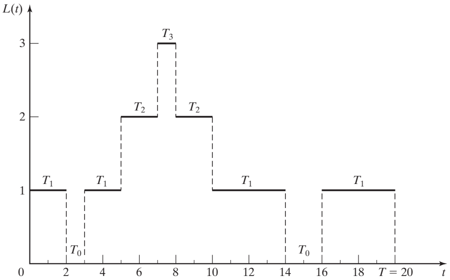

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$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$

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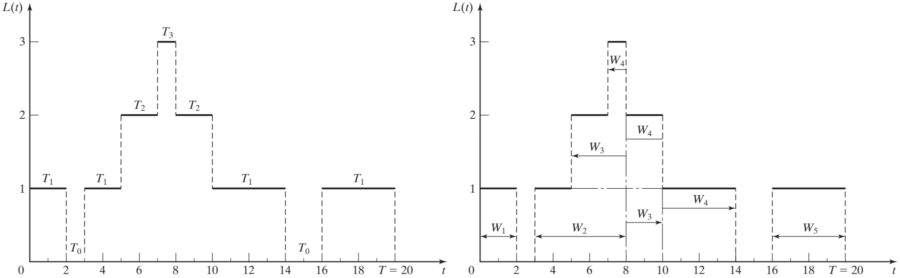

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$\widehat{W}(T) = \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i = \frac{1}{5}(2+5+5+7+4) = \frac{23}{5} = 4.6.$

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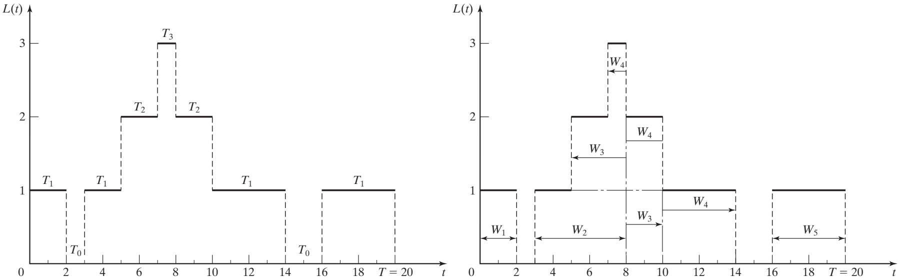

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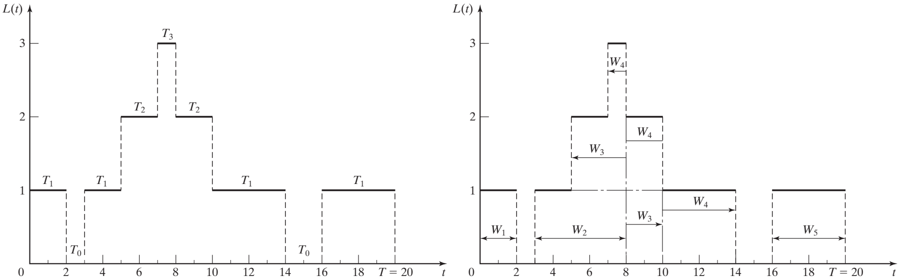

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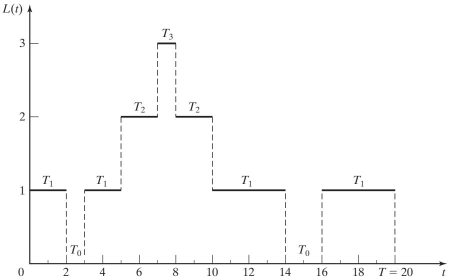

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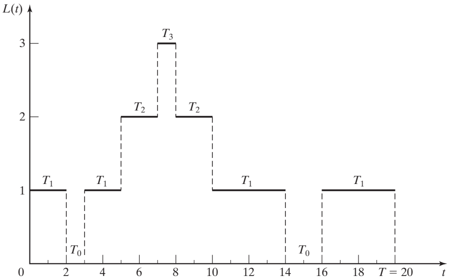

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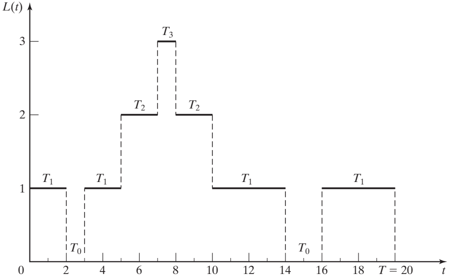

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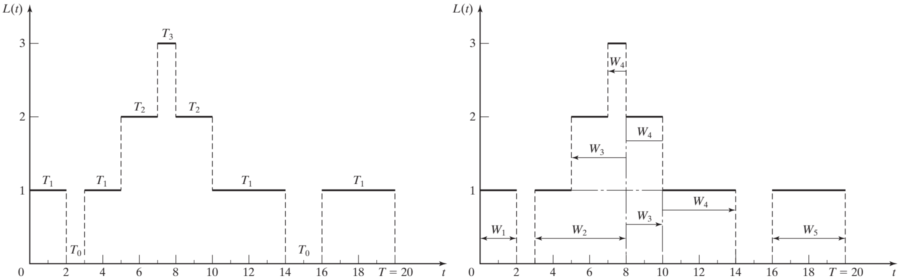

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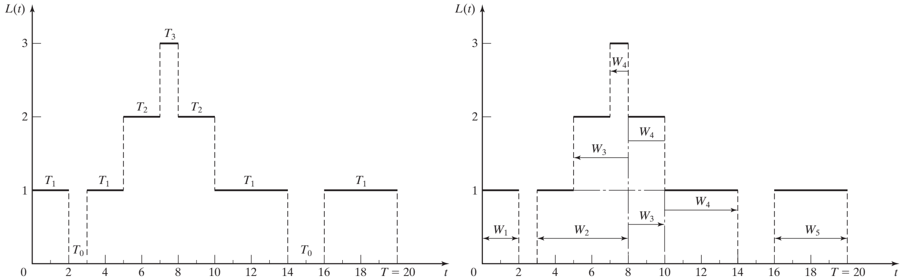

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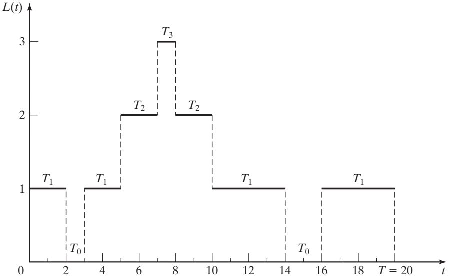

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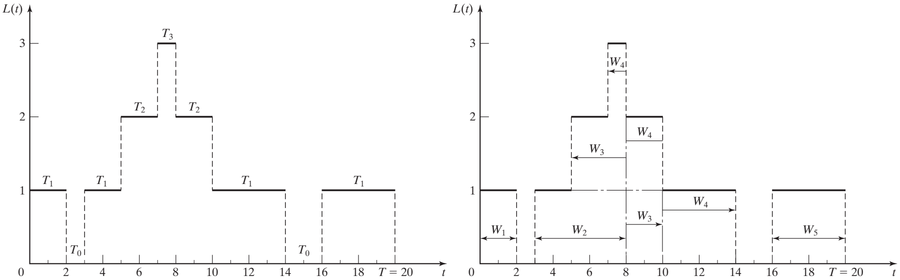

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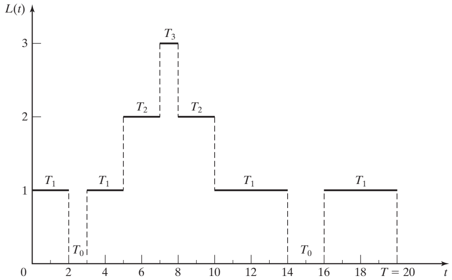

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- **The same argument for $\widehat{L}_Q(T) = \widehat{\lambda}\widehat{W}_Q(T)$.**

Theorem 3 (Little's Law – Limit/Expectation Version)

For a stable queue, let λ^* denote the arrival rate or entering rate, then

$$L = \lambda^* W, \quad L_Q = \lambda^* W_Q.$$

Caution: When λ^* is the arrival rate, the time average (W, W_Q) is based on all customers (who enter the station or are lost); When λ^* is the entering rate, the time average is only based on the customers who enters the station.

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- Some Remarks:
  - For a customer who is lost (due to the finite capacity), he spends 0 amount of time in the station (or queue).
  - Once we know anyone of L, W, L_Q and W_Q, we can compute the rest using Little's Law.

- $M/M/1$ Queue†
  - The interarrival times are iid random variables with $\mathrm{Exp}(\lambda)$ distribution, that is to say, *customers arrive according to a Poisson process with rate* λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/M/1$ queue is stable **if and only if** $\lambda < \mu$.
  - Due to unlimited capacity, arrival rate $=$ entering rate.

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- We now want to compute all the measures P_n, L, W, L_Q and W_Q.

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- Recall that L can be computed via $L = \sum_{n=0}^{\infty} nP_n$, where P_n has two interpretations:
  - Long-run proportion of time that the station contains exactly n customers;
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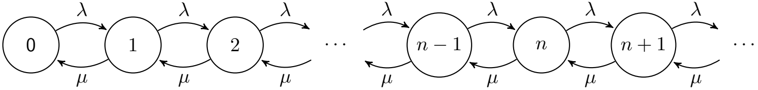
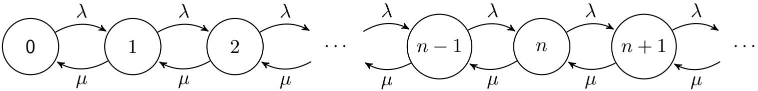
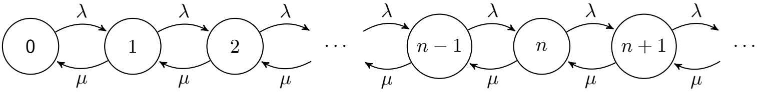
- The state space diagram is as follows:

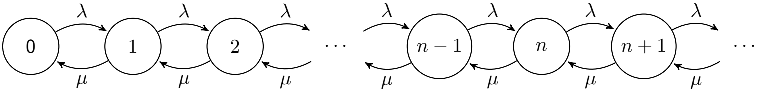



0
1
2
$\cdots$
$n-1$
n
$n+1$
$\cdots$
λ
μ



Key Observation 1

Rate at which the process leaves state n
$=$ Rate at which the process enters state n.



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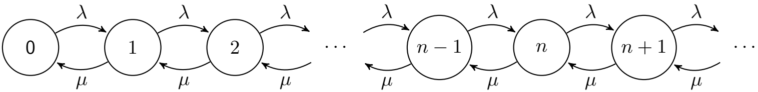
Heuristic Proof.

- In any time interval, the number of transitions into state n must equal to within 1 the number of transitions out of state n. (Why?)
- Hence, in the long run, the rate into state n must equal the rate out of state n.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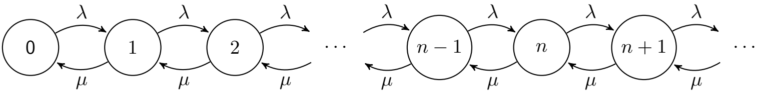

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Rate at which the process leaves state $0 = P_0\lambda$;
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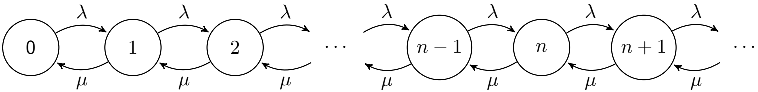

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Fact

If $X_1, \ldots, X_n$ are independent random variables, and $X_i \sim \mathrm{Exp}(\lambda_i)$, $i = 1, \ldots, n$, then

$$\min\{X_1, \ldots, X_n\} \sim \mathrm{Exp}(\lambda_1 + \cdots + \lambda_n).$$

Theorem 4 (Limiting Distribution of $M/M/1$ Queue)

For an $M/M/1$ queue, when it is stable ($\lambda < \mu$), its limiting (steady-state) distribution is given by

$$P_n = (1-\rho)\rho^n, \quad n \geq 0,$$

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Proof. Due to Observations 1 & 2,

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Rewriting these equations gives

$$\begin{aligned} P_0\lambda &= P_1\mu, \\ P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1. \end{aligned}$$

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$$P_0 = 1 - \rho, \quad \text{and} \quad P_n = (1-\rho)\rho^n, \quad n \geq 1. \quad \blacksquare$$

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.

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- $M/M/s$ Queue†
    - Customers arrive according to a Poisson process with rate λ.
    - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
    - There are s parallel servers.
    - The customers form a single queue and get served by the next available server in an FCFS fashion.
    - The capacity is unlimited, i.e., waiting space is unlimited.
    - $M/M/s$ queue is stable **if and only if** $\lambda < s\mu$.
    - Due to unlimited capacity, arrival rate $=$ entering rate.

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  - Due to unlimited capacity, arrival rate $=$ entering rate.
- $M/M/s$ queue is a generalized version of $M/M/1$ queue. Let $s = 1$, all results should degenerate to those of $M/M/1$.

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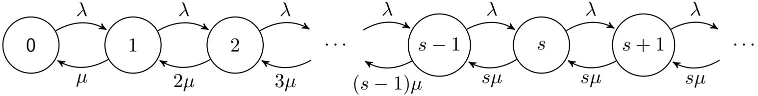
- The state space diagram is as follows:



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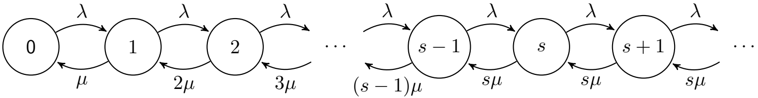

Theorem 5 (Limiting Distribution of $M/M/s$ Queue)

For an $M/M/s$ queue, when it is stable ($\lambda < s\mu$), its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \frac{s^s}{s!}\frac{\rho^{s+1}}{1-\rho}\right]^{-1} \rho_n \ , \quad n \geq 0,$$

where the *server utilization* $\rho := \lambda/(s\mu) < 1$, and

$$\rho_n := \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, & \text{if } 0 \leq n \leq s, \\ \frac{s^s}{s!}\rho^n, & \text{if } n \geq s+1. \end{cases}$$

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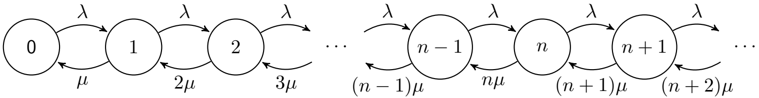
- Or, one can still derive P_n via the state space diagram:



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- $L_Q = 0$, $W_Q = 0$.

- $M/M/1/K$ Queue[†]
  - Customers arrive according to a Poisson process with rate λ.
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  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is K, $K \geq 1$, i.e., the maximal number of customers waiting in queue + customers in server $\leq K$.
  - A customer who finds the station is full (K customers there) leaves immediately (lost).
  - The entering rate, denoted as λ_e, is smaller than the arrival rate λ.
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- In steady state
  - $\mathbb{P}(\text{station is full}) = P_K$.
  - Entering rate $\lambda_e = \lambda(1 - P_K)$.

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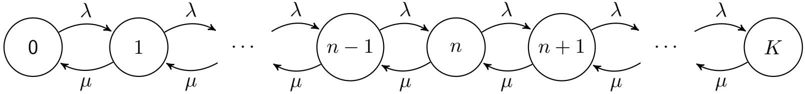

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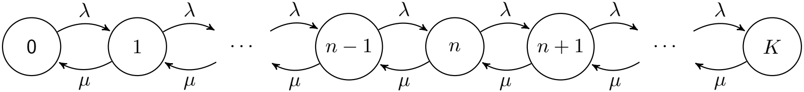
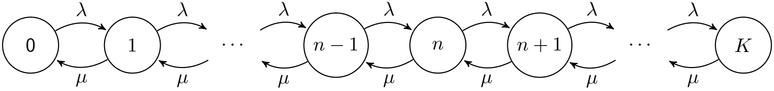


Theorem 7 (Limiting Distribution of $M/M/1/K$ Queue)

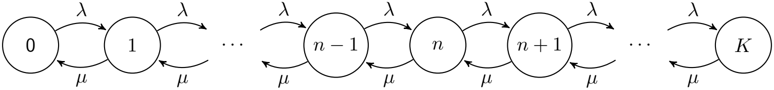
For an $M/M/1/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}, & \text{if } \rho \neq 1, \\ \frac{1}{K+1}, & \text{if } \rho = 1, \end{cases} \quad 0 \leq n \leq K,$$

where $\rho := \lambda/\mu$. (ρ is NOT the *server utilization*!)

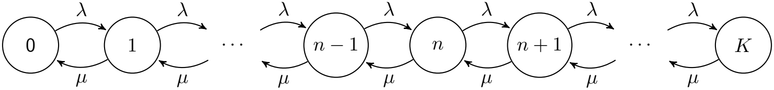


Proof.



Proof. Due to Observations 1 & 2,

State	Rate Process Leaves		Rate Process Enters
0	$P_0\lambda$	$=$	$P_1\mu$
n, $1 \le n \le K-1$	$P_n(\mu+\lambda)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$
K	$P_K\mu$	$=$	$P_{K-1}\lambda$



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K	$P_K\mu$	$=$	$P_{K-1}\lambda$

Rewriting these equations gives

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad 1 \le n \le K-1, \\
P_K\mu &= P_{K-1}\lambda.
\end{aligned}
$$

Or, equivalently,

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2, \\
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Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$P_1 = P_0\rho,$$

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- As $\rho \to \infty$, $L \to K$, $1 - P_K \to 0$, $\rho(1 - P_K) \to 1$.

- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
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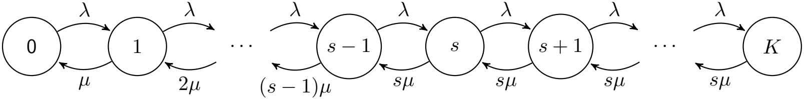
- $M/M/s/K$ queue[†] is a generalized version of $M/M/1/K$ queue. ($K \geq s$)
- The state space diagram is as follows:



- Let $s = 1$, it becomes the $M/M/1/K$ queue.
- Let $s = K$, it becomes the $M/M/K/K$ queue.
- There is no $M/M/\infty/K$ queue!

[†] $M/M/1/K$ Queue $\subset$ $M/M/s/K$ Queue $\subset$ *Birth and Death Process with Finite Capacity* $\subset$ CTMC.

Theorem 8 (Limiting Distribution of $M/M/s/K$ Queue)

For an $M/M/s/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \varrho\right]^{-1} \rho_n\ , \quad 0 \le n \le K,$$

where $\rho := \lambda/(s\mu)$, (ρ is NOT the *server utilization*!) and

$$\varrho := \begin{cases} \frac{s^s}{s!}\frac{\rho^{s+1}(1-\rho^{K-s})}{1-\rho}, & \text{if } \rho \neq 1, \\ \frac{s^s}{s!}(K-s), & \text{if } \rho = 1, \end{cases}$$

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$$\rho_n := \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, & \text{if } 0 \le n \le s, \\ \frac{s^s}{s!}\rho^n, & \text{if } s+1 \le n \le K,\ K \ge s+1. \end{cases}$$

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- The *server utilization* $= \lambda_e/(s\mu) = \rho(1 - P_K)$.

- $M/G/1$ Queue[†]
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with **arbitrary** distribution (mean: $\frac{1}{\mu}$, variance: σ^2).
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/G/1$ queue is stable **if and only if** $\lambda < \mu$.

[†] $M/G/1$ queue has an embedded discrete-time Markov chain.

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- For $M/G/\infty$, the measures are the same as those in $M/M/\infty$.

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1. Queueing Systems and Models
   - Introduction
   - Characteristics & Terminology
   - Kendall Notation
2. Poisson Process
   - Definition
   - Properties
3. Single-Station Queues
   - Notations
   - General Results
   - Little's Law
   - $M/M/1$ Queue
   - $M/M/s$ Queue
   - $M/M/\infty$ Queue
   - $M/M/1/K$ Queue
   - $M/M/s/K$ Queue
   - $M/G/1$ Queue
4. Queueing Networks
   - Jackson Networks

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Queueing Networks

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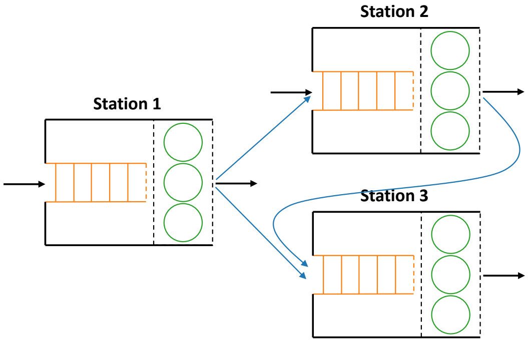

Figure: Illustration of Queueing Networks

- Jackson Queueing Network (first identified by Jackson (1963))†
  1. The network has J single-station queues.
  2. The jth station has s_j servers and a *single* queue.
  3. There is unlimited waiting space at each station (infinite capacity).
  4. Customers arrive at station j from outside according to a Poisson process with rate λ_j; all arrival processes are independent of each other.
  5. The service times at station j are iid random variables with $\mathrm{Exp}(\mu_j)$ distribution.

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7. A customer finishing service may be routed to the same station (i.e., re-enter).

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- The routing probabilities p_{ij} can be put in a matrix form as follows:

$$\boldsymbol{P} := \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1J} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2J} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{J1} & p_{J2} & p_{J3} & \cdots & p_{JJ} \end{bmatrix}.$$

- The matrix $\boldsymbol{P}$ is called the **routing matrix**.

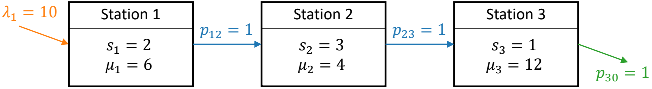
- The routing probabilities p_{ij} can be put in a matrix form as follows:

$$\boldsymbol{P} := \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1J} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2J} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{J1} & p_{J2} & p_{J3} & \cdots & p_{JJ} \end{bmatrix}.$$

- The matrix $\boldsymbol{P}$ is called the **routing matrix**.
- Since a customer leaving station i either joints some other station, or leaves, we must have

$$\sum_{j=1}^{J} p_{ij} + p_{i0} = 1, \quad 1 \le i \le J.$$

- Example 1: Tandem Queue



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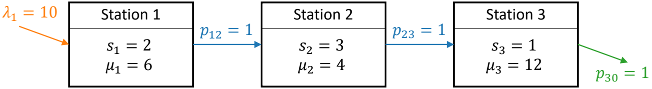

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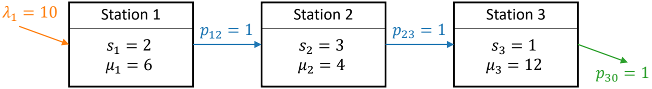

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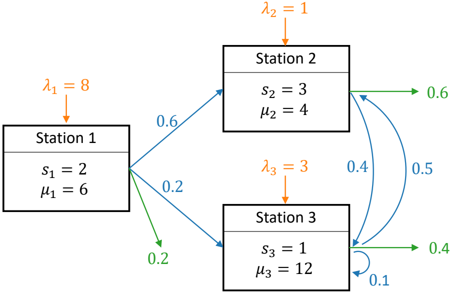

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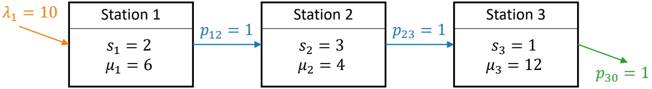

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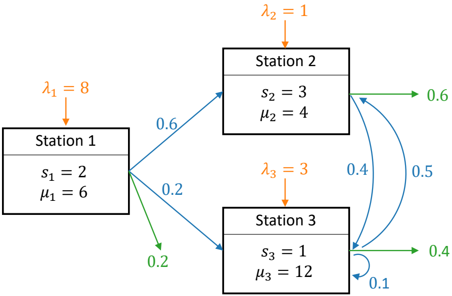

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

- Recall that customers arrive at station j from outside with rate λ_j.
- Let b_j be the rate of internal arrivals to station j.
- Then the total arrival rate to station j, denoted as a_j, is given by

$$a_j = \lambda_j + b_j, \quad 1 \le j \le J.$$

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- Hence, $b_j = \sum_{i=1}^{J} a_i p_{ij}, \quad 1 \leq j \leq J.$
- Substituting in the pervious equation, we get the **traffic equations**:
$$a_j = \lambda_j + \sum_{i=1}^{J} a_i p_{ij}, \quad 1 \leq j \leq J.$$

- Let $\boldsymbol{a}^\mathsf{T} = [a_1 \ a_2 \ \cdots \ a_J]$ and $\boldsymbol{\lambda}^\mathsf{T} = [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_J]$, the traffic equations can be written in matrix form as

$$\boldsymbol{a}^\mathsf{T} = \boldsymbol{\lambda}^\mathsf{T} + \boldsymbol{a}^\mathsf{T}\boldsymbol{P},$$

or

$$\boldsymbol{a}^\mathsf{T}(\boldsymbol{I} - \boldsymbol{P}) = \boldsymbol{\lambda}^\mathsf{T},$$

where $\boldsymbol{I}$ is the $J \times J$ identity matrix.

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- The next theorem states the stability condition for Jackson networks in terms of the above solution.

Theorem 9 (Stability of Jackson Networks)

A Jackson network with external arrival rate vector $\boldsymbol{\lambda}$ and routing matrix $\boldsymbol{P}$ is stable if:
(1) $\boldsymbol{I} - \boldsymbol{P}$ is invertible; and
(2) $a_i < s_i\mu_i$ for all $i = 1, 2, \ldots, J$, where a_i is given by the traffic equations.

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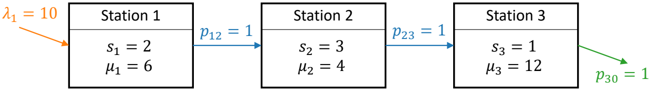

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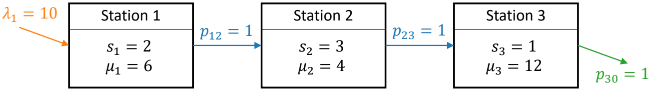
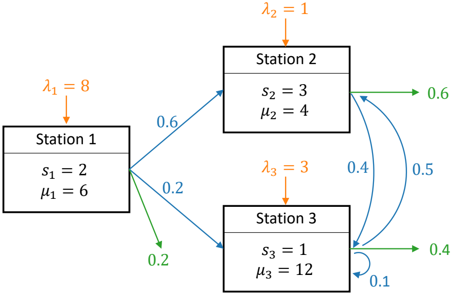


$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal(\boldsymbol{I}-\boldsymbol{P})^{-1} = [10\ 10\ 10].$$

Stable.

- Example 2: General Network



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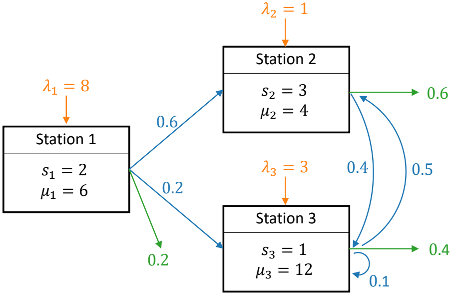

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$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal (\boldsymbol{I} - \boldsymbol{P})^{-1} = [8\ 10.7\ 9.9] \Rightarrow \text{Stable.}$$

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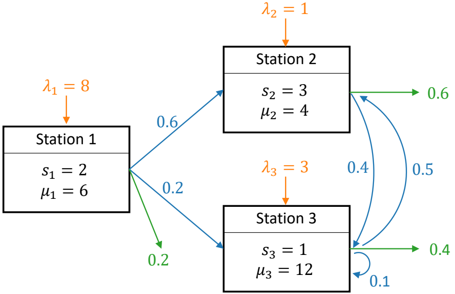

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If λ_2 is increased to 4,

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal(\boldsymbol{I} - \boldsymbol{P})^{-1} = [8\ 14.6\ 11.6] \Rightarrow \text{Unstable.}$$

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&P(n_1, n_2, \ldots, n_J) \\
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- It is a joint probability.

Theorem 10 (Limiting Distribution of Jackson Network)

For a stable Jackson network, its limiting (steady-state) distribution is given by

$$P(n_1, n_2, \ldots, n_J) = P_1(n_1)P_2(n_2)\cdots P_J(n_J),$$

for $n_j = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, J$, where $P_j(n)$ is the limiting probability that there are n customers in an $M/M/s_j$ queue with arrival rate a_j and service rate μ_j.

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- The limiting **joint** distribution of $[L_1(t), \dots, L_J(t)]$ is a **product** of the limiting **marginal** distribution of $L_j(t)$, $j = 1, \dots, J$.
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- The limiting distribution of station j is the same as that in an **isolated** $M/M/s_j$ queue with arrival rate a_j and service rate μ_j.
  (a_j's are solved from the **traffic equations**.)